

Cubic constraints for the resolvents of the ABJM matrix model and its cousins

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Abstract

A set of Schwinger-Dyson equations forming constraints for at most three resolvent functions are considered for a class of Chern-Simons matter matrix models with two nodes labelled by a non-vanishing number n . The two cases $n = 2$ and $n = -2$ label respectively the ABJM matrix model, which is the hyperbolic lift of the affine $A_1^{(1)}$ quiver matrix model, and the lens space matrix model. In the planar limit, we derive two cubic loop equations for the two planar resolvents. One of these reduces to the quadratic one when $n = \pm 2$.

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1 Introduction

Chern-Simons-matter (CSm) matrix models [1–14] have attracted considerable attentions in recent years in the context of theory on multiple M2 branes and its generalization [15–17]. They belong to a class of two matrix (to be denoted by two nodes in this paper) models connected by the measure factor which is attributed to the contributions from the n bi-fundamental multiplets upon localization of the CSm action in three dimensions. The cases $n = 2$, $n = -2$ correspond to the celebrated ABJM matrix model and the lens space matrix model respectively and are well studied [18–25] mainly by the Fermi-gas approach.

This class of matrix models is interesting also from the point of view of the q -deformation [26–48] of the Virasoro/W block [49–56] and of the 2d-4d connection [57–59] (For more references, see, for example, [60].): the $n = 1$ case provides a hyperbolic lift of the A_2 quiver hermitean matrix model that obeys W_3 constraints [61–64] and that produces [65] the $su(3)$, $N_f = 6$ Witten-Gaiotto curve [66, 67] while the $n = 2$ case provides a hyperbolic lift of the $A_m^{(1)}$ ($m = 1$) affine quiver matrix [68] model that is defined by the incidence matrix of the extended Cartan matrix and whose spacetime interpretation is yet obscure to us. A class of CSm models labelled by n provides a deformation of these cases and we study a set of Schwinger-Dyson equations forming cubic constraints from this generic point of view in this paper.

In the next section, we briefly recall the partition function of the CSm matrix model with two nodes. In section three, we consider the Schwinger-Dyson equations which take the form of the second and the third order constraints for the two resolvents. In section four, the planar limit of the equations derived in section three is taken. We derive a cubic loop equation for each of the symmetric and antisymmetric combination of the planar loop resolvents. The remarkable simplicity takes place in those cases $n = \pm 2$, where one of the two cubic equations reduces to a quadratic one. In Appendix A and B, we give some detail of the derivations of the cubic loop equations.

2 The partition function

The partition function of the Chern-Simons-matter matrix models with 2 nodes is defined by

$$Z := \int d^{N_1} u \int d^{N_2} w e^{S_{\text{eff}}}, \quad (2.1)$$

where the “effective action” S_{eff} is given by

$$\begin{aligned} e^{S_{\text{eff}}} = & \prod_{1 \leq i < j \leq N_1} \left(2 \sinh \frac{u_i - u_j}{2} \right)^2 \prod_{1 \leq a < b \leq N_2} \left(2 \sinh \frac{w_a - w_b}{2} \right)^2 \\ & \times \prod_{i=1}^{N_1} \prod_{a=1}^{N_2} \left(\cosh \frac{u_i - w_a}{2} \right)^{-n} \exp \left(-\frac{\kappa_1}{2g_s} \sum_{i=1}^{N_1} u_i^2 - \frac{\kappa_2}{2g_s} \sum_{a=1}^{N_2} w_a^2 \right). \end{aligned} \quad (2.2)$$

If we set

$$g_s = \frac{2\pi i}{k}, \quad k_1 = k \kappa_1, \quad k_2 = k \kappa_2, \quad (2.3)$$

we have

$$-\frac{\kappa_i}{2g_s} = \frac{ik_i}{4\pi}. \quad (2.4)$$

Then the partition function (2.1) arises from the localization applied to a supersymmetric $U(N_1)_{k_1} \times U(N_2)_{k_2}$ Chern-Simons theory with n bi-fundamental hypermultiplets.

In the following, we assume that $n \neq 0$. The average of a function $f(u, w)$ with respect to Z is denoted by

$$\langle f(u, w) \rangle := \frac{1}{Z} \int d^{N_1} u \int d^{N_2} w f(u, w) e^{S_{\text{eff}}}. \quad (2.5)$$

3 Constraints for resolvents

The resolvents

$$\langle \hat{\omega}_i(z) \rangle, \quad (i = 1, 2) \quad (3.1)$$

play important roles in matrix models. Here

$$\hat{\omega}_1(z) := g_s \sum_{i=1}^{N_1} \frac{1}{z - e^{u_i}}, \quad \hat{\omega}_2(z) := g_s \sum_{a=1}^{N_2} \frac{1}{z - e^{w_a}}. \quad (3.2)$$

In this section, we derive second and third order constraints for $\hat{\omega}_i(z)$. It is known that instead of (3.1), the resolvents of the following form is natural in the matrix model of Chern-Simons type:

$$\langle \hat{v}_i(z) \rangle \quad (i = 1, 2), \quad (3.3)$$

where

$$\hat{v}_1(z) := g_s \sum_{i=1}^{N_1} \frac{z + e^{u_i}}{z - e^{u_i}}, \quad \hat{v}_2(z) := g_s \sum_{a=1}^{N_2} \frac{z + e^{w_a}}{z - e^{w_a}}. \quad (3.4)$$

(3.2) and (3.4) are related by

$$\hat{v}_i(z) = 2z \hat{\omega}_i(z) - t_i. \quad (3.5)$$

Here $t_i := N_i g_s$ are the 't Hooft couplings.

For the sake of simplicity, we use $\hat{\omega}_i(z)$ in order to derive constraints for the resolvents. The constraints for $\hat{\omega}_i(z)$ are easily converted into those for $\hat{v}_i(z)$.

3.1 Second order constraints

From

$$\int d^{N_1} u \int d^{N_2} w \sum_{i=1}^{N_1} \frac{\partial}{\partial u_i} \left(\frac{1}{z - e^{u_i}} e^{S_{\text{eff}}} \right) = 0, \quad (3.6)$$

which is the hyperbolic counterpart of the Virasoro constraints [69–73], we obtain the following constraint:

$$\begin{aligned} & \left\langle \sum_i \frac{e^{u_i}}{(z - e^{u_i})^2} \right\rangle + \left\langle \sum_i \sum_{j \neq i} \frac{1}{(z - e^{u_i})} \coth \frac{u_i - u_j}{2} \right\rangle \\ & - \frac{n}{2} \left\langle \sum_i \sum_a \frac{1}{(z - e^{u_i})} \tanh \frac{u_i - w_a}{2} \right\rangle - \frac{\kappa_1}{g_s} \left\langle \sum_i \frac{u_i}{z - e^{u_i}} \right\rangle = 0. \end{aligned} \quad (3.7)$$

Using an identity

$$\sum_i \frac{e^{u_i}}{(z - e^{u_i})^2} + \sum_i \sum_{j \neq i} \frac{1}{(z - e^{u_i})} \coth \frac{u_i - u_j}{2} = z \left(\sum_i \frac{1}{z - e^{u_i}} \right)^2 - N_1 \sum_i \frac{1}{z - e^{u_i}}, \quad (3.8)$$

we can rewrite the above constraint as follows

$$z \langle \hat{\omega}_1(z)^2 \rangle - (t_1 + \kappa_1 \log z) \langle \hat{\omega}_1(z) \rangle = \frac{n}{2} \langle \hat{R}_1^{(2)}(z) \rangle - \langle \hat{F}_1(z) \rangle, \quad (3.9)$$

where

$$\hat{F}_1(z) := \kappa_1 g_s \sum_{i=1}^{N_1} \frac{\log z - u_i}{z - e^{u_i}}, \quad (3.10)$$

$$\hat{R}_1^{(2)}(z) := g_s^2 \sum_{i=1}^{N_1} \sum_{a=1}^{N_2} \frac{1}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2}. \quad (3.11)$$

Here $\log z$ takes a real value on the positive $\text{Re } z$ axis and has a cut along the negative $\text{Re } z$ axis.

Similarly, from

$$\int d^{N_1} u \int d^{N_2} w \sum_{a=1}^{N_2} \frac{\partial}{\partial w_a} \left(\frac{1}{z + e^{w_a}} e^{S_{\text{eff}}} \right) = 0, \quad (3.12)$$

we obtain

$$z \langle \hat{\omega}_2(-z)^2 \rangle + (t_2 + \kappa_2 \log(-z)) \langle \hat{\omega}_2(-z) \rangle = -\frac{n}{2} \langle \hat{R}_2^{(2)}(-z) \rangle + \langle \hat{F}_2(-z) \rangle, \quad (3.13)$$

where

$$\hat{F}_2(z) := \kappa_2 g_s \sum_{a=1}^{N_2} \frac{\log z - w_a}{z - e^{w_a}}, \quad (3.14)$$

$$\hat{R}_2^{(2)}(z) := g_s^2 \sum_{i=1}^{N_1} \sum_{a=1}^{N_2} \frac{1}{z - e^{w_a}} \tanh \frac{w_a - u_i}{2}. \quad (3.15)$$

Here $\log(-z)$ has a cut along the positive $\text{Re } z$ axis and takes real values on the negative $\text{Re } z$ axis.

By adding (3.9) and (3.13), we find the second order constraints for the resolvent operators $\hat{\omega}_i(z)$:

$$\begin{aligned} & z \langle \hat{\omega}_1(z)^2 + n \hat{\omega}_1(z) \hat{\omega}_2(-z) + \hat{\omega}_2(-z)^2 \rangle \\ & - \left(t_1 - \frac{n}{2} t_2 + \kappa_1 \log z \right) \langle \hat{\omega}_1(z) \rangle + \left(t_2 - \frac{n}{2} t_1 + \kappa_2 \log(-z) \right) \langle \hat{\omega}_2(-z) \rangle \\ & = -\langle \hat{F}_1(z) \rangle + \langle \hat{F}_2(-z) \rangle. \end{aligned} \quad (3.16)$$

Here we have used the following identity:

$$\begin{aligned} \hat{R}_1^{(2)}(z) - \hat{R}_2^{(2)}(-z) &= g_s^2 \sum_i \sum_a \left(\frac{1}{z - e^{u_i}} - \frac{1}{z + e^{w_a}} \right) \tanh \frac{u_i - w_a}{2} \\ &= g_s^2 \sum_i \sum_a \left(\frac{2z}{(z - e^{u_i})(z + e^{w_a})} - \frac{1}{z - e^{u_i}} - \frac{1}{z + e^{w_a}} \right) \\ &= -2z \hat{\omega}_1(z) \hat{\omega}_2(-z) - t_2 \hat{\omega}_1(z) + t_1 \hat{\omega}_2(-z). \end{aligned} \quad (3.17)$$

3.2 Third order constraints

There are four third order constraints.

3.2.1 Third order constraint 1

From

$$\int d^{N_1}u \int d^{N_2}w \sum_{i=1}^{N_1} \sum_{\substack{j=1 \\ (j \neq i)}}^{N_1} \frac{\partial}{\partial u_i} \left(\frac{2}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} e^{S_{\text{eff}}} \right) = 0, \quad (3.18)$$

we obtain a constraint

$$S_{11}^{(0)}(z) + g_s S_{11}^{(1)}(z) = 0, \quad (3.19)$$

where

$$\begin{aligned} S_{11}^{(0)}(z) &= \frac{8z^2}{3} \langle \hat{\omega}_1(z)^3 \rangle - 2z(2t_1 - 4g_s + \kappa_1 \log z) \langle \hat{\omega}_1(z)^2 \rangle \\ &\quad + 2(t_1 - g_s)(t_1 - 2g_s + \kappa_1 \log z) \langle \hat{\omega}_1(z) \rangle - n \langle \hat{R}_1^{(3)}(z) \rangle + 2 \langle \hat{H}_1(z) \rangle \\ &\quad + 4z^2 g_s \langle \hat{\omega}_1(z)^2 \rangle' - 2z(2t_1 - 4g_s + \kappa_1 \log z) g_s \langle \hat{\omega}_1(z) \rangle' + \frac{8z^2}{3} g_s^2 \langle \hat{\omega}_1(z) \rangle'', \end{aligned} \quad (3.20)$$

$$\begin{aligned} S_{11}^{(1)}(z) &= \left(-2z^2 \langle \hat{\omega}_1(z)^2 \rangle + 2(t_1 - g_s)z \langle \hat{\omega}_1(z) \rangle - 2z^2 g_s \langle \hat{\omega}_1(z) \rangle' \right)' \\ &\quad + 2(t_1 - g_s) \langle \hat{\omega}_1(z) \rangle + g_s^2 \left\langle \sum_i \sum_{j \neq i} \frac{1}{(z - e^{u_i})} \frac{1}{\sinh^2 \frac{u_i - u_j}{2}} \right\rangle. \end{aligned} \quad (3.21)$$

Here ' denotes the derivative with respect to z and

$$\hat{R}_1^{(3)}(z) := g_s^3 \sum_i \sum_a \sum_{j \neq i} \frac{1}{z - e^{u_i}} \tanh \frac{u_i - u_a}{2} \coth \frac{u_i - u_j}{2}, \quad (3.22)$$

$$\hat{H}_1(z) := \kappa_1 g_s^2 \sum_i \sum_{j \neq i} \frac{\log z - u_i}{z - e^{u_i}} \coth \frac{u_i - u_j}{2}. \quad (3.23)$$

See Appendix A for details.

Since we have assumed that $n \neq 0$, the third order constraint (3.19) is equivalent to the following condition

$$\begin{aligned} \langle \hat{R}_1^{(3)}(z) \rangle &= \frac{8z^2}{3n} \langle \hat{\omega}_1(z)^3 \rangle - \frac{2z}{n} (2t_1 - 4g_s + \kappa_1 \log z) \langle \hat{\omega}_1(z)^2 \rangle \\ &\quad + \frac{2}{n} (t_1 - g_s)(t_1 - 2g_s + \kappa_1 \log z) \langle \hat{\omega}_1(z) \rangle + \frac{2}{n} \langle \hat{H}_1(z) \rangle \\ &\quad + \frac{4z^2}{n} g_s \langle \hat{\omega}_1(z)^2 \rangle' - \frac{2z}{n} (2t_1 - 4g_s + \kappa_1 \log z) g_s \langle \hat{\omega}_1(z) \rangle' \\ &\quad + \frac{8z^2}{3n} g_s^2 \langle \hat{\omega}_1(z) \rangle'' + \frac{g_s}{n} S_{11}^{(1)}(z). \end{aligned} \quad (3.24)$$

3.2.2 Third order constraint 2

Similar to the case of constraint 1,

$$\int d^{N_1}u \int d^{N_2}w \sum_{a=1}^{N_2} \sum_{\substack{b=1 \\ (b \neq a)}}^{N_2} \frac{\partial}{\partial w_a} \left(\frac{2}{z + e^{w_a}} \coth \frac{w_a - w_b}{2} e^{S_{\text{eff}}} \right) = 0 \quad (3.25)$$

implies the following condition

$$\begin{aligned} \langle \hat{R}_2^{(3)}(-z) \rangle &= \frac{8z^2}{3n} \langle \hat{\omega}_2(-z)^3 \rangle + \frac{2z}{n} (2t_2 - 4g_s + \kappa_2 \log(-z)) \langle \hat{\omega}_2(-z)^2 \rangle \\ &\quad + \frac{2}{n} (t_2 - g_s) (t_2 - 2g_s + \kappa_2 \log(-z)) \langle \hat{\omega}_2(-z) \rangle + \frac{2}{n} \langle \hat{H}_2(-z) \rangle \\ &\quad - \frac{4z^2}{n} g_s \langle \hat{\omega}_2(-z)^2 \rangle' - \frac{2z}{n} (2t_2 - 4g_s + \kappa_2 \log(-z)) g_s \langle \hat{\omega}_2(-z) \rangle' \\ &\quad + \frac{8z^2}{3n} g_s^2 \langle \hat{\omega}_2(-z) \rangle'' + \frac{g_s}{n} S_{22}^{(1)}(-z), \end{aligned} \quad (3.26)$$

where

$$\begin{aligned} S_{22}^{(1)}(z) &:= \left(-2z^2 \langle \hat{\omega}_2(z)^2 \rangle + 2(t_2 - g_s) z \langle \hat{\omega}_2(z) \rangle - 2z^2 g_s \langle \hat{\omega}_2(z) \rangle' \right)' \\ &\quad + 2(t_2 - g_s) \langle \hat{\omega}_2(z) \rangle + g_s^2 \left\langle \sum_a \sum_{b \neq a} \frac{1}{(z - e^{w_a})} \frac{1}{\sinh^2 \frac{w_a - w_b}{2}} \right\rangle. \end{aligned} \quad (3.27)$$

Here

$$\hat{R}_2^{(3)}(z) := g_s^3 \sum_i \sum_a \sum_{b \neq a} \frac{1}{z - e^{w_a}} \tanh \frac{w_a - u_i}{2} \coth \frac{w_a - w_b}{2}, \quad (3.28)$$

$$\hat{H}_2(z) := \kappa_2 g_s^2 \sum_a \sum_{b \neq a} \frac{\log z - w_a}{z - e^{w_a}} \coth \frac{w_a - w_b}{2}. \quad (3.29)$$

In our convention, ' always denotes the derivative with respect to z . Hence under $z \rightarrow -z$, it transforms as follows:

$$f(z)' = \frac{\partial}{\partial z} f(z) \longrightarrow \frac{\partial}{\partial(-z)} f(-z) = -\frac{\partial}{\partial z} f(-z) = -(f(-z))'. \quad (3.30)$$

3.2.3 Third order constraint 3

From

$$\int d^{N_1}u \int d^{N_2}w \sum_{i=1}^{N_1} \sum_{a=1}^{N_2} \frac{\partial}{\partial u_i} \left(\frac{2}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} e^{S_{\text{eff}}} \right) = 0, \quad (3.31)$$

we obtain a constraint

$$S_{12}^{(0)}(z) + g_s S_{12}^{(1)}(z) = 0, \quad (3.32)$$

where

$$\begin{aligned}
S_{12}^{(0)}(z) &= 2 \left\langle \widehat{R}_1^{(3)}(z) \right\rangle - 2n \left\langle \widehat{R}_2^{(3)}(-z) \right\rangle \\
&\quad - n \left\langle \widehat{\omega}_1(z) (2z \widehat{\omega}_2(-z) + t_2)^2 \right\rangle + 2nt_1 z \left\langle \widehat{\omega}_2(-z)^2 \right\rangle + 2nt_1(t_2 - g_s) \left\langle \widehat{\omega}_2(-z) \right\rangle \\
&\quad - 2\kappa_1 \log z \left\langle \widehat{R}_1^{(2)}(z) \right\rangle + 2 \left\langle \widehat{G}_1(z) \right\rangle \\
&\quad + 4nz^2 g_s \left\langle \widehat{\omega}_1(z) (\widehat{\omega}_2(-z))' \right\rangle + 4nz g_s \left\langle \widehat{\omega}_1(z) \widehat{\omega}_2(-z) \right\rangle \\
&\quad - 2nt_1 z g_s \left\langle \widehat{\omega}_2(-z) \right\rangle' + nt_2 g_s \left\langle \widehat{\omega}_1(z) \right\rangle,
\end{aligned} \tag{3.33}$$

$$S_{12}^{(1)}(z) = -nt_2 \left\langle \widehat{\omega}_1(z) \right\rangle - 2 \left(z \left\langle \widehat{R}_1^{(2)}(z) \right\rangle \right)' + (n+1)g_s^2 \left\langle \sum_i \sum_a \frac{1}{z - e^{u_i}} \frac{1}{\cosh^2 \frac{u_i - w_a}{2}} \right\rangle. \tag{3.34}$$

Here

$$\widehat{G}_1(z) := \kappa_1 g_s^2 \sum_i \sum_a \frac{\log z - u_i}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2}. \tag{3.35}$$

See Appendix B for details.

Using the third order constraints (3.24), (3.26) and the second order constraint (3.9), we can rewrite the terms containing $\langle \widehat{R}_1^{(3)}(z) \rangle$, $\langle \widehat{R}_2^{(3)}(-z) \rangle$ and $\langle \widehat{R}_1^{(2)}(z) \rangle$.

3.2.4 Third order constraint 4

Similar to the case of the third order constraint 3,

$$\int d^{N_1} u \int d^{N_2} w \sum_{i=1}^{N_1} \sum_{a=1}^{N_2} \frac{\partial}{\partial w_a} \left(\frac{2}{z + e^{w_a}} \tanh \frac{w_a - u_i}{2} e^{S_{\text{eff}}} \right) = 0 \tag{3.36}$$

leads to

$$S_{21}^{(0)}(-z) + g_s S_{21}^{(1)}(-z) = 0, \tag{3.37}$$

where

$$\begin{aligned}
S_{21}^{(0)}(-z) &= 2 \left\langle \widehat{R}_2^{(3)}(-z) \right\rangle - 2n \left\langle \widehat{R}_1^{(3)}(z) \right\rangle \\
&\quad - n \left\langle \widehat{\omega}_2(-z) (2z \widehat{\omega}_1(z) - t_1)^2 \right\rangle - 2nt_2 z \left\langle \widehat{\omega}_1(z)^2 \right\rangle + 2nt_2(t_1 - g_s) \left\langle \widehat{\omega}_1(z) \right\rangle \\
&\quad - 2\kappa_2 \log(-z) \left\langle \widehat{R}_2^{(2)}(-z) \right\rangle + 2 \left\langle \widehat{G}_2(-z) \right\rangle \\
&\quad - 4nz^2 g_s \left\langle \widehat{\omega}_2(-z) (\widehat{\omega}_1(z))' \right\rangle - 4nz g_s \left\langle \widehat{\omega}_2(-z) \widehat{\omega}_1(z) \right\rangle \\
&\quad - 2nt_2 z g_s \left\langle \widehat{\omega}_1(z) \right\rangle' + nt_1 g_s \left\langle \widehat{\omega}_2(-z) \right\rangle,
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
S_{21}^{(1)}(-z) &= -nt_1 \left\langle \widehat{\omega}_2(-z) \right\rangle - 2 \left(z \left\langle \widehat{R}_2^{(2)}(-z) \right\rangle \right)' \\
&\quad - (n+1)g_s^2 \left\langle \sum_i \sum_a \frac{1}{z + e^{w_a}} \frac{1}{\cosh^2 \frac{w_a - u_i}{2}} \right\rangle.
\end{aligned} \tag{3.39}$$

Here

$$\widehat{G}_2(z) := \kappa_2 g_s^2 \sum_i \sum_a \frac{\log z - w_a}{z - e^{w_a}} \tanh \frac{w_a - u_i}{2}. \quad (3.40)$$

3.2.5 Summary: third order constraints

We have obtained the following third order constraints:

$$\begin{aligned} & 2 \left\langle \widehat{R}_1^{(3)}(z) \right\rangle - 2n \left\langle \widehat{R}_2^{(3)}(-z) \right\rangle \\ & - n \left\langle \widehat{\omega}_1(z) (2z \widehat{\omega}_2(-z) + t_2)^2 \right\rangle + 2nt_1 z \left\langle \widehat{\omega}_2(-z)^2 \right\rangle + 2nt_1(t_2 - g_s) \left\langle \widehat{\omega}_2(-z) \right\rangle \\ & - 2\kappa_1 \log z \left\langle \widehat{R}_1^{(2)}(z) \right\rangle + 2 \left\langle \widehat{G}_1(z) \right\rangle \\ & + 4nz^2 g_s \left\langle \widehat{\omega}_1(z) (\widehat{\omega}_2(-z))' \right\rangle + 4nz g_s \left\langle \widehat{\omega}_1(z) \widehat{\omega}_2(-z) \right\rangle - 2nt_1 z g_s \left\langle \widehat{\omega}_2(-z) \right\rangle' \\ & - 2g_s \left(z \left\langle \widehat{R}_1^{(2)}(z) \right\rangle \right)' + (n+1)g_s^3 \left\langle \sum_i \sum_a \frac{1}{z - e^{u_i}} \frac{1}{\cosh^2 \frac{u_i - w_a}{2}} \right\rangle = 0, \end{aligned} \quad (3.41)$$

$$\begin{aligned} & 2 \left\langle \widehat{R}_2^{(3)}(-z) \right\rangle - 2n \left\langle \widehat{R}_1^{(3)}(z) \right\rangle \\ & - n \left\langle \widehat{\omega}_2(-z) (2z \widehat{\omega}_1(z) - t_1)^2 \right\rangle - 2nt_2 z \left\langle \widehat{\omega}_1(z)^2 \right\rangle + 2nt_2(t_1 - g_s) \left\langle \widehat{\omega}_1(z) \right\rangle \\ & - 2\kappa_2 \log(-z) \left\langle \widehat{R}_2^{(2)}(-z) \right\rangle + 2 \left\langle \widehat{G}_2(-z) \right\rangle \\ & - 4nz^2 g_s \left\langle \widehat{\omega}_2(-z) (\widehat{\omega}_1(z))' \right\rangle - 4nz g_s \left\langle \widehat{\omega}_2(-z) \widehat{\omega}_1(z) \right\rangle - 2nt_2 z g_s \left\langle \widehat{\omega}_1(z) \right\rangle' \\ & - 2g_s \left(z \left\langle \widehat{R}_2^{(2)}(-z) \right\rangle \right)' - (n+1)g_s^3 \left\langle \sum_i \sum_a \frac{1}{z + e^{w_a}} \frac{1}{\cosh^2 \frac{w_a - u_i}{2}} \right\rangle = 0, \end{aligned} \quad (3.42)$$

where $\langle \widehat{R}_1^{(3)}(z) \rangle$ and $\langle \widehat{R}_2^{(3)}(-z) \rangle$ are respectively given by (3.24) and (3.26). The second order constraints also imply

$$\left\langle \widehat{R}_1^{(2)}(z) \right\rangle = \frac{2z}{n} \langle \widehat{\omega}_1(z)^2 \rangle - \frac{2}{n} (t_1 + \kappa_1 \log z) \langle \widehat{\omega}_1(z) \rangle + \frac{2}{n} \left\langle \widehat{F}_1(z) \right\rangle, \quad (3.43)$$

$$\left\langle \widehat{R}_2^{(2)}(-z) \right\rangle = -\frac{2z}{n} \langle \widehat{\omega}_2(-z)^2 \rangle - \frac{2}{n} (t_2 + \kappa_2 \log(-z)) \langle \widehat{\omega}_2(-z) \rangle + \frac{2}{n} \left\langle \widehat{F}_2(-z) \right\rangle. \quad (3.44)$$

4 Planar limit and loop equations

Keeping the 't Hooft couplings $t_1 = N_1 g_s$ and $t_2 = N_2 g_s$ finite, we take the planar limit $g_s \rightarrow 0$.

For $i = 1, 2$, let

$$\omega_i(z) := \lim_{g_s \rightarrow 0} \langle \widehat{\omega}_i(z) \rangle, \quad f_i(z) := \lim_{g_s \rightarrow 0} \left\langle \widehat{F}_i(z) \right\rangle, \quad (4.1)$$

$$g_i(z) := \lim_{g_s \rightarrow 0} \left\langle \widehat{G}_i(z) \right\rangle, \quad h_i(z) := \lim_{g_s \rightarrow 0} \left\langle \widehat{H}_i(z) \right\rangle, \quad (4.2)$$

$$r_i^{(2)}(z) := \lim_{g_s \rightarrow 0} \left\langle \widehat{R}_i^{(2)}(z) \right\rangle, \quad r_i^{(3)}(z) := \lim_{g_s \rightarrow 0} \left\langle \widehat{R}_i^{(3)}(z) \right\rangle, \quad (4.3)$$

We also introduce

$$v_i(z) := \lim_{g_s \rightarrow 0} \langle \hat{v}_i(z) \rangle = 2z\omega_i(z) - t_i. \quad (4.4)$$

4.1 Planar second order constraint

The planar limit of the second order constraint (3.16) is given by

$$\begin{aligned} & z\omega_1(z)^2 + nz\omega_1(z)\omega_2(-z) + z\omega_2(-z)^2 \\ & - A_1(z)\omega_1(z) + A_2(-z)\omega_2(-z) + f_1(z) - f_2(-z) = 0, \end{aligned} \quad (4.5)$$

where

$$A_1(z) := t_1 - \frac{n}{2}t_2 + \kappa_1 \log z, \quad A_2(z) := t_2 - \frac{n}{2}t_1 + \kappa_2 \log z. \quad (4.6)$$

4.2 Planar third order constraints

The planar limit of the third order constraints (3.41) and (3.42) are respectively given by

$$\begin{aligned} & 2r_1^{(3)}(z) - 2nr_2^{(3)}(-z) - n\omega_1(z)(2z\omega_2(-z) + t_2)^2 \\ & + 2nt_1z\omega_2(-z)^2 + 2nt_1t_2\omega_2(-z) - 2\kappa_1 \log z r_1^{(2)}(z) + 2g_1(z) = 0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} & 2r_2^{(3)}(-z) - 2nr_1^{(3)}(z) - n\omega_2(-z)(2z\omega_1(z) - t_1)^2 \\ & - 2nt_2z\omega_1(z)^2 + 2nt_1t_2\omega_1(z) - 2\kappa_2 \log(-z) r_2^{(2)}(-z) + 2g_2(-z) = 0, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} r_1^{(3)}(z) &= \frac{8z^2}{3n}\omega_1(z)^3 - \frac{2z}{n}(2t_1 + \kappa_1 \log z)\omega_1(z)^2 \\ &\quad + \frac{2}{n}t_1(t_1 + \kappa_1 \log z)\omega_1(z) + \frac{2}{n}h_1(z), \\ r_2^{(3)}(-z) &= \frac{8z^2}{3n}\omega_2(-z)^3 + \frac{2z}{n}(2t_2 + \kappa_2 \log(-z))\omega_2(-z)^2 \\ &\quad + \frac{2}{n}t_2(t_2 + \kappa_2 \log(-z))\omega_2(-z) + \frac{2}{n}h_2(-z), \\ r_1^{(2)}(z) &= \frac{2z}{n}\omega_1(z)^2 - \frac{2}{n}(t_1 + \kappa_1 \log z)\omega_1(z) + \frac{2}{n}f_1(z), \\ r_2^{(2)}(-z) &= -\frac{2z}{n}\omega_2(-z)^2 - \frac{2}{n}(t_2 + \kappa_2 \log(-z))\omega_2(-z) + \frac{2}{n}f_2(-z). \end{aligned} \quad (4.9)$$

4.2.1 Explicit forms

The explicit form of the planar third order constraints (4.7) and (4.8) are respectively

$$\begin{aligned}
& \frac{16z^2}{3n}\omega_1(z)^3 - 4nz^2\omega_1(z)\omega_2(-z)^2 - \frac{16z^2}{3}\omega_2(-z)^3 \\
& - \frac{8z}{n}(t_1 + \kappa_1 \log z)\omega_1(z)^2 - 4nt_2z\omega_1(z)\omega_2(-z) - 2z(4t_2 - nt_1 + 2\kappa_2 \log(-z))\omega_2(-z)^2 \\
& + \frac{1}{n}(2t_1 - nt_2 + 2\kappa_1 \log z)(2t_1 + nt_2 + 2\kappa_1 \log z)\omega_1(z) \\
& - 2t_2(2t_2 - nt_1 + 2\kappa_1 \log(-z))\omega_2(-z) \\
& + \frac{4}{n}h_1(z) - \frac{4\kappa_1}{n}\log z f_1(z) + 2g_1(z) - 4h_2(-z) = 0,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
& - \frac{16z^2}{3}\omega_1(z)^3 - 4nz^2\omega_1(z)^2\omega_2(-z) + \frac{16z^2}{3n}\omega_2(-z)^3 \\
& + 2z(4t_1 + 2\kappa_1 \log z - nt_2)\omega_1(z)^2 + 4nt_1z\omega_1(z)\omega_2(-z) + \frac{8z}{n}(t_2 + \kappa_2 \log(-z))\omega_2(-z)^2 \\
& - 2t_1(2t_1 + 2\kappa_1 \log z - nt_2)\omega_1(z) \\
& + \frac{1}{n}(2t_2 - nt_1 + 2\kappa_2 \log(-z))(2t_2 + nt_1 + 2\kappa_2 \log(-z))\omega_2(-z) \\
& + \frac{4}{n}h_2(-z) - \frac{4\kappa_2}{n}\log(-z) f_2(-z) + 2g_2(-z) - 4h_1(z) = 0.
\end{aligned} \tag{4.11}$$

4.3 Cubic loop equations

Let

$$\omega_{\pm}(z) := \omega_1(z) \pm \omega_2(-z), \quad f_{\pm}(z) := f_1(z) \pm f_2(-z), \tag{4.12}$$

$$g_{\pm}(z) := g_1(z) \pm g_2(-z), \quad h_{\pm}(z) := h_1(z) \pm h_2(-z), \tag{4.13}$$

$$v_{\pm}(z) := v_1(z) \pm v_2(-z), \tag{4.14}$$

$$A_{\pm}(z) := A_1(z) \pm A_2(-z), \quad t_{\pm} := t_1 \pm t_2. \tag{4.15}$$

Note that

$$A_+(z) = -\frac{(n-2)}{2}t_+ + K_+(z), \quad A_-(z) = \frac{(n+2)}{2}t_- + K_-(z), \tag{4.16}$$

where

$$K_{\pm}(z) := \kappa_1 \log z \pm \kappa_2 \log(-z). \tag{4.17}$$

In terms of $\omega_{\pm}(z)$, the planar second order constraint (4.5) can be rewritten as

$$\frac{(n+2)}{2}z\omega_+(z)^2 - \frac{(n-2)}{2}z\omega_-(z)^2 - A_-(z)\omega_+(z) - A_+(z)\omega_-(z) + 2f_-(z) = 0. \tag{4.18}$$

4.3.1 Cubic equation for $\omega_+(z)$

By adding (4.10) and (4.11), we find

$$\begin{aligned}
& -\frac{(n+2)(3n-2)}{3n}z^2\omega_+(z)^3 + \left\{(n+2)t_- + \frac{(n-2)}{n}A_-(z)\right\}z\omega_+(z)^2 \\
& + \frac{1}{n}\left(A_+(z)^2 + A_-(z)^2 - 2nt_-A_-(z)\right)\omega_+(z) \\
& + \frac{1}{n}\left\{(n-2)z\omega_-(z)^2 + 2A_+(z)\omega_-(z)\right\}\left\{(n-2)z\omega_+(z) - nt_- + A_-(z)\right\} \\
& - \frac{2}{n}K_+(z)f_+(z) - \frac{2}{n}K_-(z)f_-(z) - \frac{4(n-1)}{n}h_+(z) + 2g_+(z) = 0.
\end{aligned} \tag{4.19}$$

Using the planar second order constraint (4.18), we can convert this equation into an algebraic equation for $\omega_+(z)$:

$$\begin{aligned}
& -\frac{4(n+2)}{3}z^2\omega_+(z)^3 + 4A_-(z)z\omega_+(z)^2 \\
& + \left\{A_+(z)^2 - A_-(z)^2 + 4(n-2)zf_-(z)\right\}\omega_+(z) \\
& - 2K_+(z)f_+(z) - 2\left\{(n-2)t_- - K_-(z)\right\}f_-(z) \\
& - 4(n-1)h_+(z) + 2ng_+(z) = 0.
\end{aligned} \tag{4.20}$$

For $n \neq -2$, this is a cubic equation. For $n = -2$, it is quadratic.

When $n \neq -2$, in terms of

$$x_-(z) := (n+2)z\omega_+(z) - A_-(z) = \frac{(n+2)}{2}v_-(z) - K_-(z), \tag{4.21}$$

the cubic equation (4.20) becomes

$$x_-(z)^3 - 3p(z)x_-(z) - q_+(z) = 0, \tag{4.22}$$

where

$$p(z) = \frac{1}{4}\left\{(n+2)A_+(z)^2 - (n-2)A_-(z)^2 + 4(n+2)(n-2)zf_-(z)\right\}, \tag{4.23}$$

$$\begin{aligned}
q_+(z) = & \frac{1}{4}A_-(z)\left\{3(n+2)A_+(z)^2 - (3n-2)A_-(z)^2\right\} \\
& - \frac{3(n+2)^2}{2}zK_+(z)f_+(z) + \frac{3(n+2)(3n-2)}{2}zK_-(z)f_-(z) \\
& - 3(n+2)^2(n-1)zh_+(z) + \frac{3n(n+2)^2}{2}zg_+(z).
\end{aligned} \tag{4.24}$$

4.3.2 Cubic equation for $\omega_-(z)$

By subtracting (4.11) from (4.10), we obtain

$$\begin{aligned}
& -\frac{(3n+2)(n-2)}{3n}z^2\omega_-(z)^3 + \left\{(n-2)t_+ - \frac{(n+2)}{n}A_+(z)\right\}z\omega_-(z)^2 \\
& + \frac{1}{n}\left(A_+(z)^2 + A_-(z)^2 + 2nt_+A_+(z)\right)\omega_-(z) \\
& + \frac{1}{n}\left\{(n+2)z\omega_+(z)^2 - 2A_-(z)\omega_+(z)\right\}\left\{(n+2)z\omega_-(z) - nt_+ - A_+(z)\right\} \\
& - \frac{2}{n}K_-(z)f_+(z) - \frac{2}{n}K_+(z)f_-(z) + \frac{4(n+1)}{n}h_-(z) + 2g_-(z) = 0.
\end{aligned} \tag{4.25}$$

Using the planar second order constraint (4.18), we can convert this equation into an algebraic equation for $\omega_-(z)$:

$$\begin{aligned}
& \frac{4(n-2)}{3}z^2\omega_-(z)^3 + 4A_+(z)z\omega_-(z)^2 \\
& - \left\{A_+(z)^2 - A_-(z)^2 + 4(n+2)zf_-(z)\right\}\omega_-(z) \\
& - 2K_-(z)f_+(z) + 2\left\{(n+2)t_+ + K_+(z)\right\}f_-(z) \\
& + 4(n+1)h_-(z) + 2ng_-(z) = 0.
\end{aligned} \tag{4.26}$$

For $n \neq 2$, this is a cubic equation. For $n = 2$, it is quadratic.

When $n \neq 2$, in terms of

$$x_+(z) := (n-2)z\omega_-(z) + A_+(z) = \frac{(n-2)}{2}v_+(z) + K_+(z), \tag{4.27}$$

the cubic equation (4.26) becomes

$$x_+(z)^3 - 3p(z)x_+(z) - q_-(z) = 0, \tag{4.28}$$

where $p(z)$ is given by (4.23) and

$$\begin{aligned}
q_-(z) = & -\frac{1}{4}A_+(z)\left\{(3n+2)A_+(z)^2 - 3(n-2)A_-(z)^2\right\} \\
& + \frac{3(n-2)^2}{2}zK_-(z)f_+(z) - \frac{3(n-2)(3n+2)}{2}zK_+(z)f_-(z) \\
& - 3(n-2)^2(n+1)zh_-(z) - \frac{3n(n-2)^2}{2}zg_-(z).
\end{aligned} \tag{4.29}$$

4.3.3 Remark

For $n \neq -2$, if we introduce $\alpha_-(z)$ and $\beta_-(z)$ as a solution to

$$\alpha_-(z)\beta_-(z) = p(z), \quad \alpha_-(z)^3 + \beta_-(z)^3 = q_-(z), \tag{4.30}$$

then

$$x_-(z) = \alpha_-(z) + \beta_-(z) \tag{4.31}$$

solves the cubic equation (4.22).

Similarly, for $n \neq 2$, using $\alpha_+(z)$ and $\beta_+(z)$ obeying

$$\alpha_+(z)\beta_+(z) = p(z), \quad \alpha_+(z)^3 + \beta_+(z)^3 = q_-(z), \quad (4.32)$$

we have a solution to (4.28):

$$x_+(z) = \alpha_+(z) + \beta_+(z). \quad (4.33)$$

4.4 Loop equations for special cases

4.4.1 $n = 2$ case

When $n = 2$, (4.26) reduces to a quadratic equation for $2z\omega_-(z) = v_+(z) + t_+$:

$$\begin{aligned} & K_+(z)(v_+(z) + t_+)^2 - \frac{1}{2}\{K_+(z)^2 - A_-(z)^2 + 16zf_-(z)\}(v_+(z) + t_+) \\ & - 2zK_-(z)f_+(z) + 2z\{4t_+ + K_+(z)\}f_-(z) + 12zh_-(z) + 4zg_-(z) = 0, \end{aligned} \quad (4.34)$$

where

$$A_-(z) = 2t_- + K_-(z), \quad (4.35)$$

while $x_-(z) = 2v_-(z) - K_-(z)$ obeys the cubic equation (4.22) with

$$p(z) = K_+(z)^2, \quad (4.36)$$

$$\begin{aligned} q_+(z) = & A_-(z)\{3K_+(z)^2 - A_-(z)^2\} \\ & - 24zK_+(z)f_+(z) + 24zK_-(z)f_-(z) - 48zh_+(z) + 48zg_+(z). \end{aligned} \quad (4.37)$$

Note that in the case of the ABJ(M) matrix model ($k_1 = -k_2$), (4.16) and (4.17) imply

$$K_+(z) = A_+(z) = \pi i \kappa_2. \quad (4.38)$$

4.4.2 $n = -2$ case

When $n = -2$, $x_+(z) = -2v_+(z) + K_+(z)$ obeys the cubic equation (4.28) with

$$p(z) = K_-(z)^2, \quad (4.39)$$

$$\begin{aligned} q_-(z) = & A_+(z)\{A_+(z)^2 - 3K_-(z)\}^2 \\ & + 24zK_-(z)f_+(z) - 24zK_+(z)f_-(z) + 48zh_-(z) + 48zg_-(z), \end{aligned} \quad (4.40)$$

where

$$A_+(z) = 2t_+ + K_+(z), \quad (4.41)$$

while (4.20) reduces to a quadratic equation for $2z\omega_+(z) = v_-(z) + t_-$:

$$\begin{aligned} & K_-(z)(v_-(z) + t_-)^2 + \frac{1}{2}\{A_+(z)^2 - K_-(z)^2 - 16zf_-(z)\}(v_-(z) + t_-) \\ & - 2zK_+(z)f_+(z) + 2z\{4t_- + K_-(z)\}f_-(z) + 12zh_+(z) - 4zg_+(z) = 0. \end{aligned} \quad (4.42)$$

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A Derivation of third order constraint 1: eq. (3.19)

From (3.18) we have

$$\begin{aligned}
& 2 \left\langle \sum_i \sum_{j \neq i} \frac{e^{u_i}}{(z - e^{u_i})^2} \coth \frac{u_i - u_j}{2} \right\rangle - \left\langle \sum_i \sum_{j \neq i} \frac{1}{(z - e^{u_i})} \frac{1}{\sinh^2 \frac{u_i - u_j}{2}} \right\rangle \\
& + 2 \left\langle \sum_i \sum_{j \neq i} \sum_{k \neq i} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right\rangle \\
& - n \left\langle \sum_i \sum_{j \neq i} \sum_a \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \tanh \frac{u_i - w_a}{2} \right\rangle \\
& - \frac{2\kappa_1}{g_s} \left\langle \sum_i \sum_{j \neq i} \frac{u_i}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \right\rangle = 0.
\end{aligned} \tag{A.1}$$

The third term in (A.1) can be rewritten as

$$\begin{aligned}
& 2 \left\langle \sum_i \sum_{j \neq i} \sum_{k \neq i} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right\rangle \\
& = 2 \left\langle \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right\rangle \\
& + 2 \left\langle \sum_i \sum_{j \neq i} \frac{1}{z - e^{u_i}} \coth^2 \frac{u_i - u_j}{2} \right\rangle \\
& = 2 \left\langle \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right\rangle \\
& + 2 \left\langle \sum_i \sum_{j \neq i} \frac{1}{z - e^{u_i}} \frac{1}{\sinh^2 \frac{u_i - u_j}{2}} \right\rangle + 2(N_1 - 1) \left\langle \sum_i \frac{1}{z - e^{u_i}} \right\rangle.
\end{aligned} \tag{A.2}$$

Here we have used

$$\coth^2 x = 1 + \frac{1}{\sinh^2 x}. \tag{A.3}$$

Then (A.1) can be expressed in the form (3.19) with

$$S_{11}^{(0)}(z) := 2g_s^3 \left\langle \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right\rangle \\ - n \left\langle \widehat{R}_1^{(3)}(z) \right\rangle - 2\kappa_1 g_s^2 \left\langle \sum_i \sum_{j \neq i} \frac{u_i}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \right\rangle, \quad (\text{A.4})$$

$$S_{11}^{(1)}(z) := 2g_s^2 \left\langle \sum_i \sum_{j \neq i} \frac{e^{u_i}}{(z - e^{u_i})^2} \coth \frac{u_i - u_j}{2} \right\rangle \\ + 2(N_1 - 1)g_s^2 \left\langle \sum_i \frac{1}{z - e^{u_i}} \right\rangle + g_s^2 \left\langle \sum_i \sum_{j \neq i} \frac{1}{(z - e^{u_i})} \frac{1}{\sinh^2 \frac{u_i - u_j}{2}} \right\rangle. \quad (\text{A.5})$$

In the following part, we show that $S_{11}^{(0)}(z)$ (A.4) and $S_{11}^{(1)}(z)$ (A.5) can be converted respectively into (3.20) and (3.21).

A.1 Rewriting of $S_{11}^{(0)}(z)$: from (A.4) to (3.20)

Let us rewrite the first term in the right-handed side of $S_{11}^{(0)}(z)$ (A.4). Notice that

$$2 \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \\ = \frac{2}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \left(\frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right. \\ \left. + \frac{1}{z - e^{u_j}} \coth \frac{u_j - u_k}{2} \coth \frac{u_j - u_i}{2} + \frac{1}{z - e^{u_k}} \coth \frac{u_k - u_i}{2} \coth \frac{u_k - u_j}{2} \right) \\ = \frac{2}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \frac{(z^2 + e^{u_i+u_j} + e^{u_i+u_k} + e^{u_j+u_k})}{(z - e^{u_i})(z - e^{u_j})(z - e^{u_k})} \\ = \frac{8z^2}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{(z - e^{u_i})(z - e^{u_j})(z - e^{u_k})} \\ - \frac{4z}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \left(\frac{1}{(z - e^{u_i})(z - e^{u_j})} + \frac{1}{(z - e^{u_i})(z - e^{u_k})} + \frac{1}{(z - e^{u_j})(z - e^{u_k})} \right) \\ + \frac{2}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \left(\frac{1}{z - e^{u_i}} + \frac{1}{z - e^{u_j}} + \frac{1}{z - e^{u_k}} \right) \\ = \frac{8z^2}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \frac{1}{(z - e^{u_i})(z - e^{u_j})(z - e^{u_k})} \\ - 4(N_1 - 2)z \sum_i \sum_{j \neq i} \frac{1}{(z - e^{u_i})(z - e^{u_j})} + 2(N_1 - 1)(N_1 - 2) \sum_i \frac{1}{z - e^{u_i}}. \quad (\text{A.6})$$

Also

$$\begin{aligned}
& \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \frac{1}{(z - e^{u_i})(z - e^{u_j})(z - e^{u_k})} \\
&= \sum_i \sum_j \sum_k \frac{1}{(z - e^{u_i})(z - e^{u_j})(z - e^{u_k})} \\
&- 3 \sum_i \sum_j \frac{1}{(z - e^{u_i})^2(z - e^{u_j})} + 2 \sum_i \frac{1}{(z - e^{u_i})^3} \\
&= \left(\sum_i \frac{1}{z - e^{u_i}} \right)^3 + \frac{3}{2} \left[\left(\sum_i \frac{1}{z - e^{u_i}} \right)^2 \right]' + \left(\sum_i \frac{1}{z - e^{u_i}} \right)'' .
\end{aligned} \tag{A.7}$$

Therefore,

$$\begin{aligned}
& 2 \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \\
&= \frac{8z^2}{3} \left\{ \left(\sum_i \frac{1}{z - e^{u_i}} \right)^3 + \frac{3}{2} \left[\left(\sum_i \frac{1}{z - e^{u_i}} \right)^2 \right]' + \left(\sum_i \frac{1}{z - e^{u_i}} \right)'' \right\} \\
&- 4(N_1 - 2)z \left[\left(\sum_i \frac{1}{z - e^{u_i}} \right)^2 + \left(\sum_i \frac{1}{z - e^{u_i}} \right)' \right] \\
&+ 2(N_1 - 1)(N_1 - 2) \sum_i \frac{1}{z - e^{u_i}} .
\end{aligned} \tag{A.8}$$

Thus, the first term of $S_{11}^{(0)}(z)$ (A.4) can be finally rewritten as follows

$$\begin{aligned}
& 2g_s^3 \left\langle \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \coth \frac{u_i - u_k}{2} \right\rangle \\
&= \frac{8z^2}{3} \langle \hat{\omega}_1(z)^3 \rangle - 4(t_1 - 2g_s)z \langle \hat{\omega}_1(z)^2 \rangle + 2(t_1 - g_s)(t_1 - 2g_s) \langle \hat{\omega}_1(z) \rangle \\
&+ 4z^2 g_s \langle \hat{\omega}_1(z)^2 \rangle' - 4(t_1 - 2)z g_s \langle \hat{\omega}_1(z) \rangle' + \frac{8z^2}{3} g_s^2 \langle \hat{\omega}_1(z) \rangle'' .
\end{aligned} \tag{A.9}$$

This leads to

$$\begin{aligned}
S_{11}^{(0)}(z) &= \frac{8z^2}{3} \langle \hat{\omega}_1(z)^3 \rangle - 4(t_1 - 2g_s)z \langle \hat{\omega}_1(z)^2 \rangle + 2(t_1 - g_s)(t_1 - 2g_s) \langle \hat{\omega}_1(z) \rangle \\
&+ 4z^2 g_s \langle \hat{\omega}_1(z)^2 \rangle' - 4(t_1 - 2g_s)z g_s \langle \hat{\omega}_1(z) \rangle' + \frac{8z^2}{3} g_s^2 \langle \hat{\omega}_1(z) \rangle'' \\
&- n \langle \hat{R}_1^{(3)}(z) \rangle - 2\kappa_1 g_s^2 \log z \left\langle \sum_i \sum_{j \neq i} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \right\rangle + 2 \langle \hat{H}_1(z) \rangle .
\end{aligned} \tag{A.10}$$

By substituting the following identity

$$\begin{aligned} 2g_s^2 \sum_i \sum_{j \neq i} \frac{1}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \\ = 2z\hat{\omega}_1(z)^2 - 2(t_1 - g_s)\hat{\omega}_1(z) + 2zg_s\hat{\omega}_1(z)' \end{aligned} \quad (\text{A.11})$$

into (A.10), we obtain (3.20).

A.2 Rewriting of $S_{11}^{(1)}(z)$: from (A.5) to (3.21)

Next, we rewrite $S_{11}^{(1)}(z)$. The first term in (A.5) can be rewritten as

$$2g_s^2 \left\langle \sum_i \sum_{j \neq i} \frac{e^{u_i}}{(z - e^{u_i})^2} \coth \frac{u_i - u_j}{2} \right\rangle = - \left(\left\langle \sum_i \sum_{j \neq i} \frac{2g_s^2 e^{u_i}}{(z - e^{u_i})} \coth \frac{u_i - u_j}{2} \right\rangle \right)' . \quad (\text{A.12})$$

Note that

$$\begin{aligned} & \sum_i \sum_{j \neq i} \frac{2g_s^2 e^{u_i}}{z - e^{u_i}} \coth \frac{u_i - u_j}{2} \\ &= g_s^2 \sum_i \sum_{j \neq i} \left(\frac{e^{u_i}}{z - e^{u_i}} \frac{e^{u_i} + e^{u_j}}{e^{u_i} - e^{u_j}} + \frac{e^{u_j}}{z - e^{u_j}} \frac{e^{u_j} + e^{u_i}}{e^{u_j} - e^{u_i}} \right) \\ &= g_s^2 \sum_i \sum_{j \neq i} \frac{z(e^{u_i} + e^{u_j})}{(z - e^{u_i})(z - e^{u_j})} \\ &= 2g_s^2 \sum_i \frac{ze^{u_i}}{z - e^{u_i}} \sum_{j \neq i} \frac{1}{z - e^{u_j}} \\ &= 2z^2 g_s^2 \left(\sum_i \frac{1}{z - e^{u_i}} \right)^2 - 2(N_1 - 1)zg_s^2 \sum_i \frac{1}{z - e^{u_i}} + 2z^2 g_s^2 \left(\sum_i \frac{1}{z - e^{u_i}} \right)' \\ &= 2z^2 \hat{\omega}_1(z)^2 - 2(N_1 - 1)zg_s \hat{\omega}_1(z) + 2z^2 g_s \hat{\omega}_1'(z). \end{aligned} \quad (\text{A.13})$$

Therefore, the first term in (A.5) can be written by using $\hat{\omega}_1(z)$ and its derivatives:

$$\begin{aligned} & 2g_s^2 \left\langle \sum_i \sum_{j \neq i} \frac{e^{u_i}}{(z - e^{u_i})^2} \coth \frac{u_i - u_j}{2} \right\rangle \\ &= \left(-2z^2 \langle \hat{\omega}_1(z)^2 \rangle + 2(t_1 - g_s)z \langle \hat{\omega}_1(z) \rangle - 2z^2 g_s \langle \hat{\omega}_1(z) \rangle' \right)' . \end{aligned} \quad (\text{A.14})$$

Using this relation, we can easily find the final expression (3.21).

B Derivation of third order constraint 3: eq. (3.32)

From (3.31) we have

$$\begin{aligned}
& 2 \left\langle \sum_i \sum_a \frac{e^{u_i}}{(z - e^{u_i})^2} \tanh \frac{u_i - w_a}{2} \right\rangle + \left\langle \sum_i \sum_a \frac{1}{z - e^{u_i}} \frac{1}{\cosh^2 \frac{u_i - w_a}{2}} \right\rangle \\
& + 2 \left\langle \sum_i \sum_a \sum_{j \neq i} \frac{1}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \coth \frac{u_i - u_j}{2} \right\rangle \\
& - n \left\langle \sum_i \sum_a \sum_b \frac{1}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \tanh \frac{u_i - w_b}{2} \right\rangle \\
& - \frac{2\kappa_1}{g_s} \left\langle \sum_i \sum_a \frac{u_i}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \right\rangle = 0.
\end{aligned} \tag{B.1}$$

With some work, we can rewrite this constraint in the form (3.32) with

$$\begin{aligned}
S_{12}^{(0)}(z) &:= 2 \left\langle \widehat{R}_1^{(3)}(z) \right\rangle \\
&- ng_s^3 \left\langle \sum_i \sum_a \sum_{b \neq a} \frac{1}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \tanh \frac{u_i - w_b}{2} \right\rangle \\
&- 2\kappa_1 g_s^2 \left\langle \sum_i \sum_a \frac{u_i}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \right\rangle,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
S_{12}^{(1)}(z) &:= 2g_s^2 \left\langle \sum_i \sum_a \frac{e^{u_i}}{(z - e^{u_i})^2} \tanh \frac{u_i - w_a}{2} \right\rangle \\
&+ (n+1)g_s^2 \left\langle \sum_i \sum_a \frac{1}{z - e^{u_i}} \frac{1}{\cosh^2 \frac{u_i - w_a}{2}} \right\rangle - nt_2 g_s \left\langle \sum_i \frac{1}{z - e^{u_i}} \right\rangle.
\end{aligned} \tag{B.3}$$

Here we have used

$$\tanh^2 x = 1 - \frac{1}{\cosh^2 x}. \tag{B.4}$$

In the following part, we show that $S_{12}^{(0)}(z)$ (B.2) and $S_{12}^{(1)}(z)$ (B.3) can be converted respectively into (3.33) and (3.34).

B.1 Rewriting of $S_{12}^{(0)}(z)$: from (B.2) to (3.33)

Let us rewrite the second term in $S_{12}^{(0)}(z)$ (B.2). We use the following identity

$$\begin{aligned}
& \frac{1}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \tanh \frac{u_i - w_b}{2} \\
&= \frac{1}{(z - e^{u_i})(z + e^{w_a})(z + e^{w_b})} \\
&- \frac{1}{z + e^{w_a}} \tanh \frac{w_a - u_i}{2} \coth \frac{w_a - w_b}{2} - \frac{1}{z + e^{w_b}} \tanh \frac{w_b - u_i}{2} \coth \frac{w_b - w_a}{2} \\
&- \frac{2z}{(z + e^{w_a})(z + e^{w_b})} + \frac{1}{z + e^{w_a}} + \frac{1}{z + e^{w_b}}.
\end{aligned} \tag{B.5}$$

This leads to

$$\begin{aligned}
& g_s^3 \left\langle \sum_i \sum_a \sum_{b \neq a} \frac{1}{z - e^{u_i}} \tanh \frac{u_i - w_a}{2} \tanh \frac{u_i - w_b}{2} \right\rangle \\
&= g_s^3 \left\langle \sum_i \sum_a \sum_{b \neq a} \frac{1}{(z - e^{u_i})(z + e^{w_a})(z + e^{w_b})} \right\rangle + 2 \left\langle \widehat{R}_2^{(3)}(-z) \right\rangle \\
&- 2t_1 z g_s^2 \left\langle \sum_a \sum_{b \neq a} \frac{1}{(z + e^{w_a})(z + e^{w_b})} \right\rangle + 2t_1(t_2 - g_s) \left\langle \widehat{\omega}_2(-z) \right\rangle \\
&= \left\langle \widehat{\omega}_1(z) (2z \widehat{\omega}_2(-z) + t_2)^2 \right\rangle + 2 \left\langle \widehat{R}_2^{(3)}(-z) \right\rangle - 2t_1 z \left\langle \widehat{\omega}_2(-z)^2 \right\rangle - 2t_1(t_2 - g_s) \left\langle \widehat{\omega}_2(-z) \right\rangle \\
&- 4z^2 g_s \left\langle \widehat{\omega}_1(z) (\widehat{\omega}_2(-z))' \right\rangle - 4z g_s \left\langle \widehat{\omega}_1(z) \widehat{\omega}_2(-z) \right\rangle + 2t_1 z g_s \left\langle \widehat{\omega}_2(-z) \right\rangle' - t_2 g_s \left\langle \widehat{\omega}_1(z) \right\rangle.
\end{aligned} \tag{B.6}$$

Using this relation, (B.2) is easily converted into (3.33).

B.2 Rewriting of $S_{12}^{(1)}(z)$: from (B.3) to (3.34)

The first term in (B.3) can be rewritten by using the following relation:

$$g_s^2 \left\langle \sum_i \sum_a \frac{e^{u_i}}{(z - e^{u_i})^2} \tanh \frac{u_i - w_a}{2} \right\rangle = \left(-z \left\langle \widehat{R}_1^{(2)}(z) \right\rangle \right)'. \tag{B.7}$$

With help of this, we can rewrite (B.3) in the form (3.34).

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